International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN(P): 2319-3972; ISSN(E): 2319-3980
Vol. 2, Issue 5, Nov 2013, 75-82

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

SIR MODEL WITH GENERALIZED STANDARD INCIDENCE RATE FUNCTION

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ABSTRACT

I present in this paper aims to study an SIR epidemic model with generalized saturated incidence rate function. We analyze stability of the disease-free and the endemic equilibrium is with a nonlinear incidence rate.

KEYWORDS: Epidemic SIR Model, Basic Reproduction Number R_0 , Disease-Free Equilibrium and Endemic Equilibrium, Stability

1. INTRODUCTION

Contacts between susceptibles and infectives may lead to infection, and infectives may recover at different times after they become infective. This dynamics is stochastic in nature, but, for a large population, the statistical fluctuations may be ignored and the change in the size of each compartment becomes deterministic.

Epidemic dynamics is an important method of studying the spread rules of infectious diseases qualitatively and also quantitatively. It is largely based on specific properties of population growth. Analysis trough Mathematical Modeling requires transmission rules i.e. rate of incidence.

Incidence in an epidemiological model is the rate at which susceptible become infectious. The behavior of the SIR models are greatly affected by the way in which transmission between infected and susceptible individual are modeled. Simple mass action was introduced in classical Kermack –Mcendrick model βSI, where β is transmission rate, S is susceptible population and I is infectious population [7]. Another popularly used incidence rate is standard incidence $\beta SI/N$, where N is the total population and β is daily contact rate [4]. An *SIRS* model with Saturation incidence was proposed by Lorca Mena and Hethcote [10]. Liu, Hethcote and Lewin [9] have proposed non linear incidence rate.

In the present chapter we have considered an *SIR* epidemic model with incidence rate function. If we consider nonlinear incidence rate $\frac{kI}{\rho+\beta I}$ in the model coincide with that of Pathak et.al. [12].

2. PRELIMINARIES

I. Definitions

- **Equilibria:** Given an equation $\frac{dx}{dt} = f(x)$, a point x^* is an equilibrium point if $f(x^*) = 0$.
- **Diseases free Equilibrium:** The equilibrium points of given system at the origin (0,0) are called disease free equilibrium E_0.
- **Endemic Equilibrium:** The positive solution of the given system is called endemic equilibrium $E^*(S^*, I^*, R^*)$.

- **Basic Reproduction Number (R_0):** The reproduction number R_0 is products of infection rates and durations of infection.

  i.e. Basic Reproduction Number (R_0)= (Rate of secondary infection) × (Duration of infection)

- if $R_0 < 1$ the infection cannot grow.
- if $R_0 > 1$ the disease can invade the population.

II. Some Basic SIR Models

- **Basic Sketcher of SIR Epidemic Model**

Figure 1: Basic Sketcher of SIR Model

- **First Epidemic SIR Model (Kermack-Mckendrick1927)**

Let the infected person be removed from population at a rate proportional to the current number of infective person. Assuming the former infectives enter a new class which is not susceptible to the disease. Then the equations of the model are

$$\frac{dS}{dt} = -\alpha SI$$

$$\frac{dI}{dt} = \alpha SI - \beta I$$

$$\frac{dR}{dt} = \beta I$$

Where, Total population $N=S(t)+I(t)$ $R(t)$ is constant.

- **The SIR Model with Birth and Deaths**

Using the case of measles, for example, there is an arrival of new susceptible individuals into the population. For this type of situation births and deaths must be included in the model. The following differential equations represent this model, assuming a death rate d and birth rate equal to the death rate:

$$\frac{dS}{dt} = -\alpha SI + d\left(N - S\right)$$

$$\frac{dI}{dt} = \alpha SI - \beta I - dI$$

$$\frac{dR}{dt} = \beta I - dI$$

3. THE BASIC MATHEMATICAL MODEL

In the present chapter we have considered an SIR epidemiological model with asymptotically homogeneous incidence rate function. Then our model under the framework of the following form:

$$\left.\begin{aligned}\frac{dS}{dt} &= b - dS - \phi + \gamma R \\ \frac{dI}{dt} &= \phi - (d+\mu)I \\ \frac{dR}{dt} &= \mu I - (d+\gamma)R\end{aligned}\right\} \tag{1}$$

where $S(t)$, $I(t)$ and $R(t)$ denote the number of susceptible, infective and recovered at time t, respectively. b is the recruitment rate of population, d is the natural death rate of population, μ is the natural recovery rate of infective, γ is the rate at which recovered individuals loss immunity and return to susceptible, ϕ is the transmission rate.

The transmission rate $\phi = \frac{kI}{\rho + \beta I}$ displays a saturation effect accounting for the fact that the number of contacts an individual reaches some maximal value due to spatial or social distribution of the population. Where, k is the proportionality constant, ρ is the positive constant ≥ 1, β is a positive parameter and kSI is the infection force of the disease.

4. MAIN RESULTS

In this section we study an SIR epidemic model to obtain properties of the equilibrium points and analyze sufficient conditions under which the equilibrium points are unique or global.

We rewrite the system (1),

$$\left.\begin{aligned}\dot{S} &= b - dS - \phi + \gamma \mathrm{R} \\ \dot{I} &= \phi - (d+\mu)\mathrm{I} \\ \dot{R} &= \mu \mathrm{I} - (d+\gamma)\mathrm{R}\end{aligned}\right\} \tag{2}$$

Because of the biological meaning of the components ($S(t)$,$R(t)$,I(t)), we focus on the model in the first octant of R^3.

We first consider the existence of equilibria of system (2). It is easy, by computations, to conclude that the system (2) has two equilibrium states: the disease-free equilibrium state $E_0 = (\frac{b}{d}, 0, 0)$ which exists for all parameter values and endemic equilibra (S^*, I^*, R^*). To find the endemic equilibra (S^*, I^*, R^*) of system (2) set $\dot{S} = \dot{I} = \dot{R} = 0$. Then, we obtain

$$S^* = \frac{\phi(\rho + \beta \mathrm{I}^*)}{k\,\mathrm{I}^*}$$

$$I^* = \frac{d\rho(d+\mu)(d+\gamma)[R_0 - 1]}{\beta d(d+\mu)(d+\gamma) + k(d+\gamma)(d+\mu) - k\mu\gamma}$$

$$R^* = \frac{\mu I^*}{(d+\gamma)}.$$

Which exists provided that the reproduction number $R_0 = \frac{bk}{d\rho(d+\mu)} > 1$ (3)

5. MATHEMATICAL ANALYSIS

Lemma 1: The plane S + I + R = b/d is a manifold of system (2) which is attracting in the first octant.

Proof: Summing up the three equations in (2) and denoting $N(t) = S(t) + I(t) + R(t)$, we have

$$\frac{dN}{dt} = b - dN. \quad (4)$$

It is clear that $N(t) = b/d$ is a solution of system (4) and for any $N(t') \geq 0$, the general solution of system (4) is obtain by solving system (4).

System (4) is the linear differential equation of first order so the general solution of system (4) is

$$N = \frac{1}{d}[b - (b - dN(t'))e^{-d(t-t')}].$$

Thus $\lim_{t\to\infty} N(t) = \frac{b}{d}$

This implies the conclusion.

It is clear that the limit set of system (2) is on the plane $S + I + R = b/d$. Thus, we focus on the reduced system

$$\left.\begin{aligned} \frac{dI}{dt} &= \frac{kdI(\frac{b}{d} - I - R)}{\rho + \beta I} - (d+\mu)I \equiv P(I,R) \\ \frac{dR}{dt} &= \mu I - (d+\gamma)R \equiv Q(I,R) \end{aligned}\right\} \quad (5)$$

Theorem 1: System (5) does not have nontrivial periodic orbits if $\beta(2d + \gamma + \mu) > 0.$

Proof: Consider system (5) for $I > 0$ and $R > 0$. Take a Dulac function Wiggins [14].

$$D(I,R) = \phi^{-1}$$

Notice that

$$\frac{\partial(DP)}{\partial R} + \frac{\partial(DQ)}{\partial R} = -1 - \frac{\rho(d+\gamma)}{kI} - \frac{1}{k}[\beta(2d+\gamma+\mu)]$$

$$\frac{\partial(DP)}{\partial R} + \frac{\partial(DQ)}{\partial R} < 0, \text{ If } \beta(2d+\gamma+\mu) > 0.$$

Hence, the conclusion follows.

In order to study the properties of the disease-free equilibrium E_0 and the endemic equilibrium E^*, we rescale (5) by

$$x = \frac{k}{d+\gamma} I, \qquad y = \frac{k}{d+\gamma} R, \qquad \tau = (d+\gamma)t \quad .$$

$$\begin{cases} \dfrac{dx}{d\tau} = \dfrac{px}{1+qx}(A-x-y) - mx \\ \dfrac{dy}{d\tau} = sx - y \end{cases} \tag{6}$$

where

$$p = \rho^{-1},\ q = \frac{\beta(d+\gamma)}{k(\rho d + \alpha b)},\ A = \frac{bk}{d(d+\gamma)},\ m = \frac{(d+\mu)}{(d+\gamma)} \text{ and } s = \frac{\mu}{(d+\gamma)}.$$

The trivial equilibrium (0, 0) of system (6) is the disease-free equilibrium of model (2) and the unique positive equilibrium $\left(x^*, y^*\right)$ of system (6) is the endemic equilibrium E^* of model (2) if and only if $A - m > 0$and

q > 0, where

$$x^* = \frac{Ap - m}{p(1+s) + mq} \quad y^* = sx^*.$$

We first determine the stability and topological type of (0, 0). The Jacobian matrix of system (6) at (0, 0) is

$$M_0 = \begin{pmatrix} Ap - m & 0 \\ s & -1 \end{pmatrix}$$

If $Ap - m = 0$, then there exists a small neighborhood N_0 of (0,0) such that the dynamics of system (6) are equivalent to that of

$$\frac{dx}{d\tau} = -x^2 + o((x,y))^2$$

$$\frac{dy}{d\tau} = sx - y$$

Theorem 2: If $m - Ap > 0$ the disease-free equilibrium (0, 0) of system (6) is stable hyperbolic node, $m - Ap = 0$ then saddle-node and $m - Ap < 0$ then hyperbolic saddle node.

When $m - Ap < 0$, we discuss the stability and topological type of the endemic equilibrium $\left(x^*, y^*\right)$.

The Jacobian matrix of the system (6) at $\left(x^*, y^*\right)$ is

$$M_1 = \begin{pmatrix} \dfrac{px^*[qsx^* - (Aq+1)]}{(1+qx^*)^2} & \dfrac{px^*[-(1+qx^*)]}{(1+qx^*)^2} \\ s & -1 \end{pmatrix}$$

We have that

$$\det(M_1) = \frac{px^*[Aq + (1+s)]}{(1+qx^*)^2}$$

Since $q > 0$, it follows that $\det(M_1) > 0$ and $\left(x^*, y^*\right)$ is a node or a focus or a center. Furthermore, we have the following result on the stability of $\left(x^*, y^*\right)$.

Theorem 3: Suppose $m - A < 0$, then there is a unique endemic equilibrium $\left(x^*, y^*\right)$ of model (6) which is a stable node.

Proof: $\operatorname{tr}(M_1) = \frac{px^*(qsx^* - (Aq+1)) + (1+qx^*)^2}{(1+qx^*)^2}$

The sign of $\operatorname{tr}(M_1)$ is determined by

$$S_1 = px^*(qsx^* - (Aq+1))$$

Substituting $x^* = \frac{Ap - m}{(1+s)+mq}$ into S_1 and using a straightforward calculation, we have

$$S_1 = \frac{p(Ap-m)}{[p(1+s)+mq]^2}[-Aq(p+mq) - ((s+1)(mq+p)]$$

Since $q > 0$, $[p(1+s) + mq] > 0$

and $[-Aq(p+mq) - ((mq+p)(s+1))] < 0$, hence, $S_1 < 0$. However, when $m - Ap < 0$, we have $\operatorname{tr}(M_1) < 0$. This completes the proof.

6. CONCLUSIONS

Modeling results are helpful to predict the developing tendency of disease. The model we have discussed provides learning about the transmission rate effects. In this paper, we consider an SIR model with generalized incidence rate ϕ We have derived a basic reproduction number R_0 with the special kind of transmission rule and showed that the global stability of the endemic equilibrium point $E^* = (S^*, I^*, R^*)$ depends on the basic reproduction number R_0 Our result shows that when $R_0 < 1$ the disease free equilibrium $E_0 = (b/d, 0, 0)$ is globally attractive in the first octant. When $R_0 > 1$ the endemic equilibrium E^* exists and is globally stable in the interior. Results and parametric conditions help to develop social consciousness about the disease among susceptible.

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